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Recent Developments in Finite Energy (Topological) Monopole Theory.*

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Abstract

Recent activity in the study of static, finite-energy, topologically charged monopole systems is reviewed. The main developments have been the study of axially symmetric systems, the proof of existence of static multi-monopole configurations for large monopole seperations, and the explicit construction of single monopoles of arbitrary charge and of multi-monopole configurations for small monopole seperations.

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Introduction.

According to current views the fundamental physical interactions are described by unified gauge theory. Since this theory differs from Maxwell-Lorentz theory in that it may be non-abelian and spontaneously broken it is natural to look for properties of the theory that are consequences of these new features. The two most striking properties that have been discovered so far are asymptotic freedom (1) and the existence of stable finite energy (or finite action) field configurations (2). These properties have turned out to be of interest not only in their own right but also in connexion with quark confinement (3).

In the present lectures we shall consider the stable, static, finite energy configurations in three space dimensions. The identification of such configurations with magnetic monopoles, their relevance for confinement, and the explicit construction of spherically symmetric solutions have all been discussed some time ago (3), but until recently no progress was made in the construction, or even proof of existence, of configurations which are not spherically symmetric. In the past year or so, however, some dramatic progress has been made in this direction and it is this progress which will be reviewed.

As mentioned in the abstract, the main developments have concerned the questions of existence of static monopole configurations, the construction of non-spherically symmetric monopoles (of magnetic charge greater than unity) and the study of axisymmetric systems. All these developments are based on the reduction of the usual second-order field equations to a first-order system known as the Bogomolny (B) equations (4).

Accordingly, the review will commence with a sketch of the earlier developments and the introduction of the B-equations. The sketch will include the simplification of the equations which takes place in Yang's R-gauge (5), and their subsequent linearization by means of Bäcklund transformations [6]. The review then goes on to sketch one of the major recent developments, namely, an existence proof for static separated monopole configurations due to Taubes [7]. The axisymmetric configurations are then discussed. The most striking results here are (i) the fact that axisymmetric systems can not describe separated monopoles of the Taubes kind, but only single monopoles of arbitrary charge [8] (ii) the existence of a master-potential for all the invariants [9] and (iii) the equivalence of axisymmetric B-equations to the Ernst equation of General relativity [10]. We then come to the second major development of recent times, namely, the explicit construction of solutions which describes single (axisymmetric) monopoles of arbitrary strength[11][12]. The properties of such solutions and the problem of establishing their regularity for higher monopole strengths is discussed. In one of these constructions (due to Ward) use is made of the vector-bundle formalism of Atiyah and Ward, [13] and because of its wider importance, this formalism is described in some detail. Finally, a recent construction by Ward of a solution which describes two slightly seperated monopoles, and a proposal to extend this construction to the n-monopole case, are described.

2. Early Developments

Let $(\vec{A},\Phi)$ be a static magnetic SU(2) Yang-Mills-Higgs system, with the Higgs field in the adjoint representation, and Hamiltonian

$$H = \frac{1}{2}\int d^3x \left\{ B^2 + (D\Phi)^2 + 2V(\Phi) \right\}, \tag{2.1}$$

where

$$\vec{B} = \vec{\nabla}\times\vec{A} + \tfrac{1}{2}\vec{A}\times_{\wedge}\vec{A}, \quad \vec{D}\Phi = \vec{\nabla}\Phi + \vec{A}\wedge\Phi, \quad V = \lambda(\Phi^2-c^2)^2 \geq 0, \tag{2.2}$$

and wedge denotes SU(2) outer product. The corresponding field equations are

$$\vec{D}\cdot\vec{B} = 0, \qquad \vec{D}\times\vec{B} = \Phi_\wedge\vec{D}\Phi, \qquad D^2\Phi = \partial V/\partial\Phi, \tag{2.3}$$

and by finite-energy configurations are meant solutions of (2.3) which make the integral (2.1) converge. The reason that the system admits such solutions is that the spontaneously broken potential admits (indeed requires) the boundary condition

$$\lim_{r\to\infty} \Phi(r,\theta,\varphi) \to c\,\phi(\theta,\varphi), \qquad \phi^2 = 1, \tag{2.4}$$

and the topology of SU(2) is such that $\phi(\theta,\varphi)$ may be non-trivial. More precisely, $\phi(\theta,\varphi)$ is a mapping from S_2(space) to S_2 (isospace) and such mappings fall into discrete homotopy classes labelled by an integer n. The boundary functions $\phi(\theta,\varphi)$ which belong to the non-trivial homotopy classes $n \neq 0$ (e.g. $\phi = (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi)$) can not be gauge-transformed to the trivial function $\phi = 1$, and generate the non-trivial finite energy solutions.

An important operator in this respect is the topological charge, defined as

$$Q = \frac{1}{4\pi}\int d\Omega\,(\phi, \partial_\varphi\phi_\wedge\partial_\theta\phi) \tag{2.5}$$

Q is gauge-invariant and takes the value n when $\phi(\theta,\varphi)$ belongs to the nth homotopy class [14][15]. Thus it acts as a Casimir label for the homotopy classes.

Q is also a superselection operator (commutes with all the fields) and hence it guarantees the stability of different homotopy sectors. The generalization of (2.5) for arbitrary volumes R is

$$Q = \int_R d^3x\,(\vec{\nabla}\cdot\vec{j}) \quad \text{where} \quad j_r = \varepsilon_{rst}(\phi,\partial_s\phi\wedge\partial_t\phi), \quad \phi = \Phi/|\Phi| , \qquad (2.6)$$

and since the divergence is identically zero when ϕ is regular we see that the charge is actually located only at those points where the Higgs field $\vec{\Phi}$ is zero.

By completing the square in (2 1) and carrying out some partial integrations[4] one may write H in the form

$$H = \tfrac{1}{2}\int d^3x\,\{(B - D\Phi)^2 + 2V\} \quad + \quad Q . \qquad (2.7)$$

which shows that Q also provides a lower bound for the energy.

The first non-trivial solution of the system (2.2) was found by t'Hooft,[3] who assumed spherical symmetry and thus reduced the field equations to two non-linear, coupled, but ordinary equations for the norms of the Higgs and gauge fields H(r) and K(r). He then used numerical methods to establish that these equations almost certainly had a solution and to approximate it. Later a rigorous proof for the existence of these solutions was given[16] and it was shown[17] that they must be real analytic functions of r. Finally, in the special case in which the potential V is set equal to zero (but the boundary condition (2.4) is retained) it was shown[18] that the solutions took the elementary form

$$H(r) = c\left(\frac{\cosh cr}{\sinh cr} - \frac{1}{cr}\right), \qquad K(r) = \frac{cr}{\sinh cr} . \qquad (2.8)$$

The spherically symmetric solutions so obtained turned out to have unit topological charge ($n = \pm 1$) and it was soon shown[19] that this was the only possible charge which could be obtained from spherically symmetric configurations. (Conversely, systems with unit topological charge had to be spherically symmetric[20]).

3. The Bogomolny Equation and its Linearization.

According to the results just mentioned, monopoles or monopole systems of charge greater than unity can only be obtained by dropping the assumption of spherical symmetry. But then the field equations (2.2) become extremely complicated, and the problem would probably have remained intractable had it not been noticed[4] that in the limit when the potential V is zero, the field equations reduce to the much simpler first-order set

$$\vec{B} = \vec{D}\Phi . \qquad (3.1)$$

This follows from the fact that, for fixed Q, eqn. (3.1) manifestly minimizes the energy in (2.6). The equations (3.1) are called the Bogomolny equations, and they also have a geometrical significance, because if we identify Φ with the 4th component A_4 of a 4-dimensional Euclidean gauge-potential they can be regarded as the static version of the 4-dimensional self-dual equations

$$F_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}, \quad \text{where} \quad F_{ij} = \tfrac{1}{2}\varepsilon_{ijk}B_k \quad \text{and} \quad F_{i4} = D_i\Phi . \qquad (3.2)$$

The Bogomolny system (3.1) not only has the mathematical advantages just discussed, but it has also the advantage that it is the only case in which one can hope to describe a system of separated monopoles in static equilibrium, as well as single monopoles of arbitrary strength. In fact, unless the Higgs field is in the adjoint representation, the potential is zero and the monopole charges have the same sign, there are even long-range forces between the monopoles. On the other hand, if these three conditions and the Bogomolny equation are satisfied, the components of the stress-tensor, and hence the forces between the mpnopoles vanish everywhere[21].

The Bogomolny system (3.1) contains nine equations for the twelve functions $(\vec{A}, \Phi)$, three functions remaining undetermined due to the gauge freedom. However, by using a construction of Yang for the self-dual form (3.2), they can be reduced to three equations for three unknown functions[5]. In fact, by choosing a suitable gauge (the R-gauge) and partially solving the self-dual equations, Yang reduced them to

$$\Box \ln f = \frac{1}{f^2}(\partial e \cdot \partial g), \tag{3.3}$$

and

$$\partial_{\bar{u}}\left(\frac{\partial_u e}{f^2}\right) + \partial_v\left(\frac{\partial_{\bar{v}} e}{f^2}\right) = 0, \qquad \text{where} \quad \begin{matrix} u = x + iy \\ v = z + i x_4 \end{matrix}, \tag{3.4}$$
$$\partial_u\left(\frac{\partial_{\bar{u}} g}{f^2}\right) + \partial_{\bar{v}}\left(\frac{\partial_v g}{f^2}\right) = 0,$$

$$A_\mu = -\frac{1}{2f}\begin{bmatrix} \eta^3_{\mu\nu}\partial_\nu f & \eta^-_{\mu\nu}\partial_\nu e \\ \eta^+_{\mu\nu}\partial_\nu g & -\eta^3_{\mu\nu}\partial_\nu f \end{bmatrix}, \quad \text{and} \quad \begin{matrix} \eta^a_{\mu\nu} = \varepsilon_{4a\mu\nu} + \delta_{a\mu}\delta_{\nu 4} - \delta_{a\nu}\delta_{\mu 4}, \\ \eta^\pm = \eta_1 \pm i\eta_2 . \end{matrix} \tag{3.5}$$

Later Corrigan[6] et al. (CFGY) observed that the Yang equations (3.4) were invariant with respect to the involutive transformations

$$\text{I} : \quad \begin{bmatrix} e_I & f_I \\ f_I & g_I \end{bmatrix} = \begin{bmatrix} e & f \\ f & g \end{bmatrix}^{-1}, \tag{3.6}$$

and

$$\text{B} : \quad f^b = \frac{1}{f}, \quad \begin{matrix} (e^b_u, g^b_v, e^b_{\bar{v}}, g^b_{\bar{u}})/f^b \\ = (-g_v, e_u, g_{\bar{u}}, -e_{\bar{v}})/f \end{matrix} \quad \text{where} \quad e_u = \partial_u e \quad \text{etc.} \tag{3.7}$$

In fact eqns. (3.4) are just the integrability conditions for (3.7). The transformation (3.6) is actually a gauge-transformation, but the transformation (3.7) is not. In fact, (3.7) is a Bäcklund transformation i.e. is a first order differential transformation connecting different solutions of a second order differential equation.

Using the (non-involutive) product IB, Corrigan et al.[6] were able to obtain at least one class of solutions of the Yang equations by linearizing them as follows:

First let $\Delta_0(x)$ be any solution of the d'Alembertian equation. Then

$$f = e = g = \Delta_0^{-1} \qquad \text{where} \qquad \Box\Delta_0 = 0 \tag{3.8}$$

is a solution of the Yang equations. Next, let $\Delta_r(x)$ for $r = 0, \pm 1, \pm 2, \ldots$ be any set of functions satisfying the Cauchy-Riemann-like equations

$$\frac{\partial \Delta_r}{\partial \bar{u}} = -\frac{\partial \Delta_{r+1}}{\partial \bar{v}}, \qquad \frac{\partial \Delta_r}{\partial v} = \frac{\partial \Delta_{r+1}}{\partial u}, \tag{3.9}$$

(which imply the d'Alembertian equation) and for any integer $k \geqslant 1$ form the matrix

$$D(k) = \begin{bmatrix} \Delta_{k-1} & \Delta_{k-2} & \cdot & \Delta_1 & \Delta_0 \\ \Delta_{k-2} & \Delta_{k-3} & & \Delta_0 & \Delta_{-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Delta_1 & \Delta_0 & & \Delta_{3-k} & \Delta_{2-k} \\ \Delta_0 & \Delta_{-1} & \cdot & \Delta_{2-k} & \Delta_{1-k} \end{bmatrix} \tag{3.10}$$

Then, provided that the matrix D(k) is invertible, the corner elements of the inverse matrix

$$D^{-1}(k) = \begin{bmatrix} e & \cdot & \cdot & \cdot & f \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f & \cdot & \cdot & \cdot & g \end{bmatrix}, \tag{3.11}$$

form a solution of the Yang system. Furthermore, the Bäcklund transformations IB and BI connect the k with the $k \pm 1$ solutions. In particular, each family of solutions is generated from the $(k = 1)$ solution Δ_0 of the d'Alembertian equation. However, the reality and singularity properties are not preserved by the Bäcklund transformations, so the choice of Δ_0 is not trivial.

We shall refer to a set of functions Δ_r satisfying (3.9) as a Durham string, and it is perhaps worth mentioning at this point that each Durham string can be generated by a potential function as follows: let

$$\Delta = \sum_{-\infty}^{\infty} \Delta_r e^{ir\psi} \tag{3.12}$$

be the Fourier transform of the Δ_r. Then the Durham string equation (3.9) is just the condition that Δ should depend on only three of the five variables (x, ψ), namely,

$$\Delta = \Delta(e^{i\psi}, \mu, \nu) \tag{3.13}$$

where (in cylindrical coordinates)

$$\mu = ix_4 + z + \rho e^{i(\varphi+\psi)} , \qquad \nu = ix_4 - z\, e^{-i(\varphi+\psi)} . \tag{3.14}$$

Thus any function of $(e^{i\psi}, \mu, \nu)$ (with a Fourier transform) may be used to generate a Durham string and hence a solution of the self-dual and Bogomolny equations. The deeper reason for this result will be seen when we consider the AW construction in section 7.

An interesting result, due to Prasad [22], is that the norm of the Higgs field is directly related to the determinant of the Durham matrix D(k), by the formula

$$\Phi^2 = c^2 - \Delta \ln\big(\det D(k)\big) . \tag{3.15}$$

The proof of this formula, which uses the Bäcklund transformations IB, is given in Appendix A.

All of the recent advances in monopole theory have been made using the Bogomolny system (3.1) and hence, from now on only this system will be considered. We shall see that it admits solutions describing both single monopoles of arbitrary strength and separated monopoles. It should be noted, however, that in the more realistic case when the potential is not set equal to zero, the seperated monopole configuration becomes unstable. Thus for the seperated solutions the Bogomolny condition is crucial, whereas for the solutions describing single monopoles of arbitrary strength it is probably only a technical device that allows us to obtain the solutions in closed form.

4. The Existence of Static Multi-Monopole Systems.

The first important recent development in the theory of monopoles, and perhaps the most fundamental development, is a proof, due to Taubes [7], that the Bogomolny equations (3.1) do indeed admit static, separated, monopole configurations. Here the monopoles are supposed to be separated by distances rather larger than the monopole "core", the latter being defined as the region outside of which the two "massive" gauge fields fall-off exponentially, leaving only the magnetic and Higgs fields with long-range ($1/r$) components. (One recalls that for a Hamiltonian of the form (2.1) two of the three gauge fields acquire masses because of the Higgs-Kibble mechanism).

The idea in Taubes proof is to start from an initial approximate configuration $(\vec{A}_0, \Phi_0)$ and, by successive iteration of the Bogomolny equation, to construct a sequence $(\vec{A}_s, \Phi_s)$ which converges to an exact solution as $s \to \infty$. It is clear that the basic problems are

(i) to construct a function space for $(\vec{A}, \Phi)$ in which convergence makes sense and is reasonably likely.

(ii) to choose the initial configuration sufficiently close to the expected solution for the sequence to have a reasonable chance of converging.

The function space chosen by Taubes was the Sobelev space

$$\int_R d^3x \left\{ (\nabla_i A_j, \nabla_i A_j) + (\nabla_i \Phi, \nabla_i \Phi) + (A_i, A_i) + (\Phi, \Phi) \right\} < \infty \tag{4.1}$$

(for every finite volume R). A comparison of (4.1) with the Hamiltonian (2.1) shows that there is a close connection between the configurations $(\vec{A}, \Phi)$ which lie in (4.1) and the configurations with finite-energy, and this is what makes (4.1) a natural choice. However, (in contrast to the case of vortices and spherically symmetric monopoles) the exact equivalence of (4.1) with finite energy has not been established. (Note that (4.1) is actually gauge-dependent).

In order to construct an initial configuration Taubes divided the Euclidean 3-space E(3) into three regions - the expected monopole cores of radius C^{-1}, shells of thickness C^{-1} surrounding the cores, and an 'exterior' region consisting of the rest of E(3). He then chose as initial configuration

(a) the exact 1-monopole solutions of section 2 inside the cores

(b) an exact 'exterior' soluton of the Bogomolny equations in the exterior region

(c) Some C^{∞} transition functions, to connect these two sets of solutions smoothly, in the shells.

The exact 'exterior' solutions are actually solutions of the Maxwell-Higgs U(1) subsystem of SU(2), up to some topological factors which are chosen to produce unit charge within each core[21] (Taubes actually trivialized these factors by using a singular (Dirac) gauge).

Using the properties of the Bogomolny system, in particular the fact that it is an elliptic system, Taubes was able to show that the iteration of this initial configuration does indeed converge in the norm (4.1). Furthermore, he showed that the exact solutions had to be real analytic. The actual proof is far from trivial (it runs to fifty typed pages) and constitutes a major tour de force.

5. Axially Symmetric Configurations.

With the question of existence of multi-monopole solutions settled by Taubes result the emphasis shifts to the question of single monopoles of arbitrary strength and to the explicit construction of solutions. It is not obvious, of course, that such solutions exist or can be constructed in terms of elementary functions but experience with the spherically symmetric solution (2.7) and the geometrical nature of the Bogomolny equations suggests these possibilities. A natural first step towards such a construction is to consider axially symmetric configurations, since these are the next simplest to spherically symmetric ones. Axisymmetric systems are characterized[24](8) by the fact that they admit a smooth isovector $\omega(x)$ which implements rotations around the axis of symmetry,

$$D_{\varphi}\,\Phi(x) = \omega(x)\wedge\Phi(x), \qquad (5.1)$$

where φ is the azimuthal angle. Further, since (5.1) should hold not only for $\Phi(x)$ but also for all of its covariant derivatives, it is easy to see that $\omega(x)$ must satisfy the integrability condition

$$D_i\,\omega(x) = \varepsilon_{i\varphi j}\,B_j(x) \quad . \qquad (5.2)$$

Eqns. (5.1) and (5.2) characterize the axial symmetry.

At first sight one should expect the axisymmetric solutions to describe both single monopoles of arbitrary strength and colinear multi-monopole systems, in particular to describe two monopoles. But here one encounters a major surprise. It turns out[8] that axisymmetric systems can describe only single monopoles!

The proof of this rather surprising result runs as follows: First, because Φ is subharmonic and $|\Phi| \to C \neq 0$ at infinity, the zeros of Φ and hence the topological charge can only lie at points or on one-dimensional curves. Next, from the definition (2.5) of the topological charge (restricted to an arbitrary volume V that intersects the z-axis at two points z_1 and z_2)

and from (5.1), it follows after some algebra that the charge contained in the volume V is just

$$\Delta Q = \ell(z_1) - \ell(z_2) \quad \text{where} \quad \ell = (\omega, \phi) \quad \text{and} \quad \phi = \bar{\Phi}/|\bar{\Phi}| . \tag{5.3}$$

By letting z_1 and z_2 coincide we can see first of all that there is no charge located <u>off</u> the axis. Next from (5.1) and (5.2) we have (for smooth $(\vec{A}, \bar{\Phi})$)

$$\omega_\wedge \bar{\Phi} = 0 \qquad \text{and} \qquad \omega^2 = \text{constant}, \tag{5.4}$$

on the z-axis. Combining (5.3) and (5.4) we then have

$$\ell^2 = \text{constant, and hence} \quad \Delta Q = 0, \ z \ \text{constant}, \tag{5.5}$$

on the axis. For like charges (which as we have seen, is the only stable case) equation (5.5) implies that all the charge must be concentrated at a single point, as required.

Although the result just obtained restricts axial symmetry to single monopoles, such systems are still of interest, so we consider now some further properties of axial symmetry. The first such result is that axi-symmetric systems admit a master-potential W(x) from which the gauge-invariants $\bar{\Phi}^2$, ω^2 and $(\omega, \bar{\Phi})$ may be obtained by differentiation [5]. For if we take the inner-product of (5.1) with $\bar{\Phi}$ and ω respectively, we obtain the Cauchy-Riemann equations

$$\partial_i \omega^2 - \rho^2 \partial_i \bar{\Phi}^2 = 2 \varepsilon_{i\varphi j} \partial_j (\omega, \bar{\Phi}) \tag{5.6}$$

and it is easy to see that the content of these equations is precisely that there should exist a scalar W(x) such that

$$\frac{\partial W}{\partial z} = (\omega, \bar{\Phi}), \quad 2\rho \frac{\partial W}{\partial \rho} = \omega^2 - \rho^2(\bar{\Phi}^2 - c^2), \quad \Delta W = (\bar{\Phi}^2 - c^2) \tag{5.7}$$

where z and ρ are the usual cylindrical coordinates.

By comparing (5.7) with (3.15) we obtain also a remarkable connection between the master-potential and the Durham determinant $\det \mathcal{D}(k)$, namely,

$$W(x) = -\ln[\det \mathcal{D}(k)] \tag{5.8}$$

Equation (5.8) actually follows from (3.15) only up to a harmonic function. But by a proof analogous to that used to establish (3.15), the harmonic function can be shown to be at most a constant (see Appendix A).

Another interesting property of the axisymmetric case is the simplification obtained for the Bogomolny equations. However, it should be noted that since axial symmetry is abelian, it cannot reduce the number of functions in the Bogomolny equations but only determine their φ-dependence. On the other hand, axial symmetry can be supplemented by <u>mirror</u>-symmetry (symmetry under reflexions in any plane through the axis) and then the number of functions in the Bogomolny equations is reduced from twelve to six, including one gauge function. The six gauge fields take the form

$$\begin{aligned} \bar{\Phi} &= (\phi_1, \phi_2, 0) & A_\rho &= (0, 0, W_\rho) \\ A_\varphi &= (\eta_1, \eta_2, 0) & A_z &= (0, 0, W_z) \end{aligned}, \tag{5.9}$$

and since all the fields are φ-independent, the system has then effectively been reduced to a Maxwell, or U(1), system, with one gauge-potential $\vec{W}$, and two real two-component Higgs fields η and ϕ. It is then not surprising that the Bogomolny equations reduce to the U(1)-covariant system

$$\begin{aligned} \mathcal{D}_\rho \phi &= \rho^{-1} \mathcal{D}_z \eta \\ \mathcal{D}_z \phi &= -\rho^{-1} \mathcal{D}_\rho \eta \\ \nabla \times W &= \rho^{-1} (\eta_\wedge \phi) \end{aligned} \tag{5.10}$$

where D denotes U(1) covariant derivative, and wedge the 2-dimensional outer product. (The factor ρ^{-1} occurs because the coordinates (ρ , z) are curvilinear). The system (5.10) was first proposed (without using mirror symmetry) by Manton [9] and by Jang, Park and Wali [23] and is sometimes known as the Manton Ansatz. The system consists of five equations for the six unknown functions (5.9) and it can be partially solved in two different ways, as follows:

I The four equations (a) (b) can be solved, and the solution turns out to be just the master-potential solution (5.6). This shows that <u>all</u> the gauge-invariants can be derived from the master-potential in the mirror symmetric case. The remaining equation (c) can then be written [9] in the form

$$\Delta h = (D\phi)^2 h \qquad \text{where} \qquad h = |\Phi| \ , \quad \phi = \Phi / h \ , \tag{5.11}$$

and is an implicit (4th order) equation for the master-potential.

II The three equations (a)(c) or (b)(c) can be solved and in a suitable gauge the solution is

$$\begin{aligned} (\phi_1, \phi_2) &= f^{-1}(g_{,z}, -f_{,z}) \ , & W_\rho &= \phi_3 \ , \\ (\eta_1, \eta_2) &= \rho f^{-1}(g_{,\rho}, -f_{,\rho}) \ , & W_z &= f^{-1} \eta_1 \end{aligned} \tag{5.12}$$

where f and g are two unknown functions. When the solution (5.12) is inserted in the remaining two equations the latter reduce to an equation of the form[10]

$$(\mathrm{Re}\, \varepsilon)\, \Delta \varepsilon = (\vec{\nabla} \varepsilon)^2 \qquad \text{where} \qquad \varepsilon = f + ig \tag{5.13}$$

This is a remarkable result because, as those who are familiar with General Relativity will recognize, (5.10) is just the Ernst equation for axially symmetric gravitational fields! Thus the axisymmetric Bogomolny equations for monopoles and the Ernst equations for gravitation are the same!

In order to see whether the axisymmetric system actually admits solutions of monopole charge greater than unity, a numerical analysis for n = 2,3 was carried out independently by Adler[25] and by Rebbi and Rossi[26]. The numerical analysis suggested strongly that solutions exist and, rather surprisingly, they indicated that the highest concentration of energy would not be at the centre of the monopole. This result is confirmed by the beautiful graphs for n = 2,3,4,5 shown by Palla at this conference. Palla's result shows that the maximum energy density is situated in a torus which coincides roughly with the edge of the monopole core.

The most remarkable result for the axisymmetric case, however, is that <u>exact</u> solutions of the axisymmetric field equations (5.10) have now been found. They have been found independently by Palla et al. using the Ernst equation (5.13) and by Ward. As the Palla construction will be described by Dr. Palla in his seminar I shall consider here only the Ward construction. Although this construction yields the same solution as Palla's, it is quite different, and, as we shall see, it can be generalized to the non-axisymmetric case.

6. The Atiyah-Ward Construction: Mini-Model.

The Ward construction[11] of the n = 2 monopole solutions is based on the use of vector-bundles to construct self-dual fields. Originally the vector-bundles were intended[13] for the case of instantons, but the method has turned out to be more useful for monopoles. To illustrate the basic AW idea we consider first a mini-model in real 2-dimensional Euclidean space E(2).

The straight lines on E(2) are of the conventional form

$$\alpha x + \beta y + \gamma = 0 \quad , \tag{6.1}$$

and hence are parametrized by the variables (α, β, γ) modulo a common factor δ where δ is any real number. Thus they form the 2-dimensional (compact) projective space P(2). It is not possible to cover P(2) with a single system of 2-dimensional coordinates, but nevertheless it will be useful to use the conventional 2-dimensional system (m,c) obtained by neglecting the vertical lines and writing (6.1) in the form

$$y = mx + c \quad . \tag{6.2}$$

The AW idea is a *generalization* of the fact that the scalar fields $\phi(m,c)$ on P(2) and the abelian gauge-fields B(x) on E(2) can be considered as transforms of one another in the following sense: Let B(x) be any abelian gauge-field on E(2) and (m,c) any line in P(2). Then the transform $\phi(m,c)$ of F(x) is defined to be

$$\phi(m,c) = \int_\Delta dx\,dy\, B(x,y) \quad , \tag{6.3}$$

where Δ is the triangle enclosed by the line y(m,c) and the coordinate axes in E(2). Conversely, if $\phi(m,c)$ has the correct analyticity properties for $\phi(m, y-mx)$ to have a Laurent expansion in m, then the inverse of (6.3) is

$$B(x,y) = \frac{1}{2\pi i}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\oint \frac{dm}{m}\,\phi(m, y-mx) \tag{6.4}$$

where the integration is around the unit circle in complex m-space.

Of course, since E(2) is 2-dimensional B(xy) has only one component so that (6.3) and (6.4) is a transform between scalars on P(2) and E(2) (rather similar to, but not identical with, the Radon transform). However, it is convenient to think of B(xy) as a gauge-field, because of the later generalization and because, as we shall see, the gauge-potential $\vec{A}(xy)$ for B(xy) enters naturally in the proof of (6.4) from (6.3).

To establish (6.4) we first write $B(x) = \vec{\nabla}\times\vec{A}$ and then convert (6.3) into an integral around the perimeter of the triangle to obtain

$$\phi(m,c) = \oint A_k\, dx^k \,, \qquad k=1,2. \tag{6.5}$$

Next we let (xy) be any point on the line (m,c)(between the vertices of the triangle) and split the integral into

$$\phi = \phi_+ + \phi_- \qquad \text{where} \qquad \phi_+ = \oint_{(0,0)}^{(x,y)} \,, \quad \phi_- = \oint_{(x,y)}^{(0,0)} \tag{6.6}$$

each integral being taken anti-clockwise along the perimeter. Differentiating along the line (m,c) we then have

$$A_x + m A_y = (\partial_x + m\partial_y)\phi_+ = -(\partial_x + m\partial_y)\phi_- \quad . \tag{6.7}$$

But since the vertices of the triangle (c/m,o) and (o,c) we see that if $\phi(m, y-mx)$ is analytic for $m \neq 0,\infty$ then ϕ_+ is analytic for $m \neq \infty$ and ϕ_- for $m \neq 0$. Hence A_x and A_y are the leading terms in the expansions of $\phi_\pm$ respectively, and we have

$$A_x = \frac{1}{2\pi i}\int \frac{dm}{m}\,\partial_x \phi_+ \qquad \text{and} \qquad A_y = \frac{-1}{2\pi i}\int \frac{dm}{m}\,\partial_y \phi_- \quad . \tag{6.8}$$

Equation (6.4) then follows immediately from the definition $\vec{B} = \vec{\nabla}\times\vec{A}$. Note that the gauge freedom $A_k \to A_k + \partial_k \Lambda$ corresponds exactly to the freedom of assigning the m-independent term in the expansion of $\phi(m,c)$ to ϕ_+ or ϕ_-, indeed $\Lambda = \overset{\circ}{\phi}_+ - \overset{\circ}{\phi}_-$ where superscript zero denotes the zero-order term in the Taylor expansions of ϕ_+ and ϕ_- respectively.

One might now ask the question : what is the connection between the transformation $\phi(m,c) \longleftrightarrow F(xy)$ and fibre-bundles? The point is that since P(2) is not completely covered by the coordinates (m,c), it is natural to think of (m,c) as section of a fibre-bundle with P(2) as base space and fibres of one dimension. Then the mini-model may be interpreted as a relationship between sections of fibre bundles over P(2) and gauge fields on E(2).

7. The Atiyah-Ward Construction: Maxi-Model.

The actual AW construction (maxi-model) consists in lifting the relationship between functions over P(2) and E(2) of the mini-model to a similar relation between functions over CP(3) and C(4), where CP(3) denotes complex projective 3-space and C(4) denotes complex Euclidean space. However, in the maxi case the constraints on the functions are much tighter - the functions on C(4) must be self-dual and those on CP(3) (which are $SL(2,C)$ -valued) must possess a generalized Laurent expansion (to be defined below).

The space E(2) is replaced by C(4) by introducing 4 complex coordinates x_μ. The 'lines' in this space (the analogues of (6.1)) are then the complex 2-planes

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} ix_4 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & ix_4 - x_3 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}, \qquad (7.1)$$

where (ω_1, ω_2) and (π_1, π_2) are complex 2-spinors. It is clear that the complex 2-planes are parametrized by the four complex numbers $(\omega_1, \omega_2, \pi_1, \pi_2)$ modulo λ, where λ is any complex number. Hence the complex 2-plane form the (3-complex-dimensional) space CP(3). Although CP(3) is compact and cannot be completely spanned by three complex coordinates, it is convenient, nevertheless, to introduce a 3-dimensional coordinate system

$$(\mu, \nu, \zeta) = (\omega_1/\pi_1 \, , \; \omega_2/\pi_2 \, , \; \pi_2/\pi_1) \, . \qquad (7.2)$$

Such a coordinate system omits the planes $\pi_1 = 0$ and $\pi_2 = 0$ and reduce the planes (7.1) to

$$\begin{aligned} \mu &= ix_4 + x_3 + (x_1 + ix_2)\zeta \, , \\ \nu &= ix_4 - x_3 + (x_1 - ix_2)\zeta^{-1} \, . \end{aligned} \qquad (7.3)$$

Note that if we identify ix_4 with t and ζ with $e^{i\varphi}$, the functions μ, ν in (7.3) agree with those in (3.14) . Note also that the variables (μ, ν, ζ) are just the well-known 'twistor' variables introduced by Penrose in the context of General Relativity.

To obtain the analogue of equation (6.3) of the mini-model we first note that the "coordinate-axes" are

$$x_1 - ix_2 = x_3 + ix_4 = 0 \qquad \text{and} \qquad x_1 + ix_2 = x_3 - ix_4 = 0 , \tag{7.4}$$

respectively, and that for fixed (μ, ν, ζ) the plane (7.3) intersects them at

$$R(\mu,\nu,\zeta) = \tfrac{1}{2}(\nu\zeta, i\nu\zeta, \mu, -i\mu) \qquad \text{and} \qquad Q(\mu,\nu,\zeta') = \tfrac{1}{2}(\mu\zeta', -i\mu\zeta', -\nu, -i\nu) \tag{7.5}$$

respectively. Then for any gauge-potential $A_\mu(x)$ over C(4) one forms the function

$$g(\mu,\nu,\zeta) = P\, e^{\int_R^Q A_\mu(x) dx^\mu} , \tag{7.6}$$

where the integral is to be taken along a path from R to Q within the plane (7.3), and P denotes path-ordering. Equation (7.6) is the required analogue of (6.3) (the integration along the axes being omitted for later convenience). Note that $g(\mu,\nu,\zeta)$ depends only on (μ, ν, ζ) and is SL(2,C)-valued.

It is precisely in (7.6) that the self-duality enters. The point is that since (7.6) can be along any path in the plane the function $g(\mu,\nu,\zeta)$ will be uniquely defined, if, and only if, the integral is path-independent, and this will be so if, and only if, the field $F_{\mu\nu}(x)$ for $A_\mu(x)$ is <u>self-dual</u>. The reason is simple: the planes (7.3) are <u>anti-self-dual</u> in the sense that

$$dx_\pi = dx'_\pi = 0 \quad \Leftrightarrow \quad dx_{(\mu} dx'_{\nu)} + \varepsilon_{\mu\nu\lambda\sigma} dx_\lambda dx_\sigma = 0 , \tag{7.7}$$

(see appendix B) and hence the projections $F_{\mu\nu} dx_\mu dx'_\nu$ will be zero if, and only if, $F_{\mu\nu}$ is self-dual. Thus for C(4) only self-dual fields admit the transformation into functions over CP(3).

The converse is also true - not every SL(2,c)-valued function over CP(3) admits the transformation into self-dual fields. To see this, choose any point x_μ in the plane (7.3) and make the decomposition

$$g(\mu,\nu,\zeta) = g_+(\zeta,x)\, g_-(\zeta',x) \tag{7.8}$$

where

$$g_+(\zeta,x) = P\, e^{\int_Q^x A_k dx^k} \qquad \text{and} \qquad g_-(\zeta',x) = P\, e^{\int_x^R A_k dx^k} . \tag{7.9}$$

From the expressions for Q and R in (7.5) we see that g_+ and g_- are regular for $\zeta \neq \infty$ and $\zeta \neq 0$ respectively, and so (7.8) is a generalized 'Laurent' decomposition. But not every SL(2,C)-valued function over CP(3) admits such a generalized Laurent decomposition. Thus finally the transformations, $F_{\mu\nu}$ on C(4) to $g(\mu,\nu,\zeta)$ on CP(3) and back, are seen to be limited both ways - to self-dual $F_{\mu\nu}$ and to Laurent-decomposable $g(\mu,\nu,\zeta)$.

We have anticipated here the result that every Laurent decomposable $g(\mu,\nu,\zeta)$ does indeed admit a transformation to a self-dual $F_{\mu\nu}$. To see that it does, and to obtain the $F_{\mu\nu}$ field, one proceeds as in the mini-model: differentiating the decomposition (7.8) in the plane (7.3) and using the Liouville theorem one finds that

$$P_\mu\, g_+^{-1} \partial_\mu g_+ = P_\mu\, g_- \partial_\mu g_-^{-1} = P_\mu A_\mu , \quad \text{where} \quad P_\mu = \sigma_\mu \binom{1}{\zeta} \tag{7.10}$$

the σ_μ are the Pauli matrices $(i, \vec{\sigma})$, and the $A_\mu(x)$ are independent of ζ. The $A_\pm(x)$ are then the potentials for the self-dual fields.

In fibre bundle language the functions $g(\mu,\nu,\zeta)$ may be regarded as sections of an SL(2,C)-fibred bundle with CP(3) as base-space. The condition that $g(\mu,\nu,\zeta)$ is Laurent-decomposable may be regarded as the condition that $g(\mu,\nu,\zeta)$ trivializes on a line in CP(3) (i.e. for fixed x in (7.1)).

8. The Atiyah-Ward Ansatz

The AW construction translates the problem of finding solutions of the Bogomolny equation into finding functions $g(\mu,\nu,\zeta)$ which are Laurent-decomposable and which lead to the correct reality, regularity and boundary conditions for the fields $(\vec{A},\Phi)$.

The full class of Laurent-decomposable $g(\mu,\nu,\zeta)$ is not yet known, but Atiyah and Ward have pointed out that a sufficient condition for decomposability is that $g(\mu,\nu,\zeta)$ be of the form

$$g(\mu,\nu,\zeta) = \begin{bmatrix} \rho(\mu,\nu,\zeta) & \zeta^{\ell} \\ -\zeta^{-\ell} & 0 \end{bmatrix}, \tag{8.1}$$

where ℓ is an integer and ρ is an arbitrary function. Equation (8.1) is known as the AW Ansatz. In the case of instantons (which are algebraic) the Ansatz is sufficient to describe all the self-dual solutions, but it is not clear that such will be the case for monopoles. However, as we shall see, the Ansatz yields at least two important classes of solutions.

Since the AW Ansatz reduces the choice of $g(\mu,\nu,\zeta)$ to the choice of a single function $\rho(\mu,\nu,\zeta)$ one might expect to find a relation between the Ansatz and the single potential $\Delta(\mu,\nu,e^{i\psi})$ of the Durham string, and indeed there is a remarkably simple connection, namely,

$$\Delta(\mu,\nu,e^{i\psi}) = \rho(\mu,\nu,\zeta) \qquad \text{for} \quad \zeta = e^{i\psi}. \tag{8.2}$$

This is the deeper reason for the existence of the Durham potential Δ.

In terms of the AW-Ansatz the Durham string may be then written as

$$\Delta_r(x) = \frac{1}{2\pi i}\int \frac{d\zeta}{\zeta}\,\zeta^{r}\,\rho(\mu,\nu,\zeta), \tag{8.3}$$

where μ and ν are given by (7.3). The great advantage of (8.3) is that instead of having to derive the gauge-potentials from $\rho(\mu,\nu,\zeta)$ by means cumbersome decomposition formula (7.10), one can derive them from (8.3) and the algebraic formulae (3.11) and (3.5). The whole problem then reduces to finding a $\rho(\mu,\nu,\zeta)$ which leads to the correct reality, regularity and boundary conditions for the gauge fields $(\vec{A},\Phi)$.

9. The Ward Solution and its Generalization.

Although the AW Ansatz (8.1) was originally intended for instantons, it has turned out to be more useful for monopoles, where by a judicious choice of $\rho(\mu,\nu,\zeta)$, Ward[11] was able to construct the same single-monopole of charge two solution as Forgacs et al.[12] Ward's form of the solution is in a gauge which is not manifestly real or time-independent, but has the advantage that the solution can easily be generalized to monopoles of arbitrary strength and even to separated monopole solutions.

Ward's starting point was to note that for the known spherically symmetric, charge one, monopole the unknown function $\rho(\mu,\nu,\zeta)$ in the AW ansatz turned out to be just the entire function

$$\rho_1(\mu,\nu) = \frac{e^{\mu} - e^{\nu}}{\gamma}, \qquad \gamma = \frac{\mu-\nu}{2} \tag{9.1}$$

where, from (7.3),

$$\mu = (t+z) + \rho\, e^{i\varphi}\zeta, \quad \nu = (t-z) + \rho\, e^{-i\varphi}\zeta^{-1}, \quad \gamma = z + \tfrac{1}{2}\rho\,(e^{i\varphi}\zeta - e^{-i\varphi}\zeta^{-1}),$$

and, in order to simplify the notation, the Higgs constant c has been normalized to 1. For reasons which will become clearer below Ward then guessed that the generalization of (9.1) to the charge-two case would be the entire function

$$\rho_2(\mu,\nu) = \frac{e^{\mu} + e^{\nu}}{\gamma^2 + (\frac{\pi}{2})^2}. \tag{9.2}$$

For the same reasons the generalizations of (9.1) to the case of arbitrary ℓ is guessed to be the entire functions

$$\rho_\ell(\mu,\nu) = \frac{e^{\mu} + (-1)^\ell e^{\nu}}{H_\ell(\gamma)}, \tag{9.3}$$

where

$$H_\ell(\gamma) = \gamma \prod_{m=1}^{(\ell-1)/2} (\gamma^2 + m^2\pi^2) \qquad \text{and} \qquad H_\ell(\gamma) = \prod_{m=1}^{\ell/2} (\gamma^2 + (m-\tfrac{1}{2})^2\pi^2), \tag{9.4}$$

for ℓ odd and even respectively.

It is easy to see from (9.1) and (9.3) that the components of the Durham string generated by the ρ-functions in (9.3) are just :

$$\Delta_r(x) = \frac{e^{2t}}{2\pi}\int_0^{2\pi} d\psi\, e^{-\rho\cos\psi}\, \frac{\sin(\rho\sin\psi + iz)}{H_\ell(\rho\sin\psi + iz)}\, e^{ir\psi}, \tag{9.5}$$

for odd ℓ, and a similar expression with sin $\longrightarrow$ cos for even ℓ. In particular the central function Δ_0 takes the form

$$\Delta_0(x) = \sum_{-\frac{(\ell-1)}{2}}^{\frac{\ell-1}{2}} \frac{\sinh r(k)}{r(k)} \qquad \text{where} \qquad r^2(k) = \rho^2 + (z + i\pi k)^2 \tag{9.6}$$

Because of the trivial time-dependence of the Δ_r, one sees also that the d'Alembertian equation reduces to the Yukawa equation

$$\Box\, \Delta_r(x) = 0 \quad \Rightarrow \quad (\Delta - c^2)\, \Delta_r(x) = 0. \tag{9.7}$$

where the Higgs constant c has been recalled in order to emphasize its role.

Since the AW construction guarentees that the fields derived from the $\rho_\ell(\mu,\nu)$ in (9.3) satisfy the self-dual equations (3.2) what has to be checked is that the solutions satisfy the other requirements for monopole solutions of charge n. These are

(i) reality and time-independence

(ii) the boundary condition $|\Phi| \to c - \frac{n}{r}$ as $r \to \infty$ $(c=1)$

(iii) regularity.

The boundary condition (ii) is necessary and sufficient to describe a monopole of charge n because, when the Bogomolny equations are satisfied, we have from (2.1) and (2.7)

$$4\pi c\, Q = H = \int d^3x\, (D\Phi)^2 = \frac{1}{2}\int d^3x\, \Delta\,\Phi^2 = \int r^2 d\Omega\, |\Phi| \frac{\partial|\Phi|}{\partial r}. \tag{9.8}$$

In making the Ansatz (9.3) the ρ-functions have, of course, been chosen in order to satisfy the conditions (i) - (iii), and we now consider these conditions in turn. First, condition (i) requires that the AW-matrix $g(\mu,\nu,\zeta)$ be time-independent and hermitian, or be gauge-transformable to such a $g(\mu,\nu,\zeta)$.

Here, hermitian means that $g^{\dagger}(\mu,\nu,\zeta) = g(\mu,\nu,-\bar{\zeta}^{-1})$ where dagger denotes hermitian conjugate, and the existence of a gauge transformation means that there should exist a pair of invertible 2 x 2 matrices $X(\zeta^{-1})$ and $Y(\zeta)$, regular in ζ^{-1} and ζ respectively, such that

$$g(\mu,\nu,\zeta) \rightarrow \tilde{g}(\mu,\nu,\zeta) = X(\zeta^{-1})\, g(\mu,\nu,\zeta)\, Y(\zeta) \tag{9.9}$$

where $\tilde{g}$ is time-independent and hermitian. It is clear that the $g(\mu,\nu)$ constructed with the ρ-functions in (9.3) are not themselves time-independent or real, but they admit a transformation of the form (9.9), namely,

$$\begin{pmatrix} \rho & -\zeta^{\ell} \\ \zeta^{-\ell} & 0 \end{pmatrix} = \begin{pmatrix} e^{\nu/2} & 0 \\ 0 & e^{-\nu/2} \end{pmatrix} \begin{pmatrix} \frac{e^{\gamma}+(-1)^{\ell}e^{-\gamma}}{H_{\ell}(\gamma)} & e^{-\gamma}\zeta^{\ell}(-1)^{\ell} \\ e^{-\gamma}\zeta^{-\ell} & H_{\ell}(\gamma)e^{-\gamma} \end{pmatrix} \begin{pmatrix} e^{\mu/2} & -H_{\ell}(\gamma)\,\zeta^{\ell} e^{-\mu/2} \\ 0 & e^{-\mu/2} \end{pmatrix}, \tag{9.10}$$

and the transformed matrix is manifestly time-independent and real.

Actually, the matrices X and Y in (9.10) will be regular in ζ and ζ^{-1} only if the denominator function $H_{\ell}(\gamma)$ is a polynomial of degree at most ℓ, and this is why the denominators in (9.3) are chosen to be polynomials. The zeros of the polynomial are chosen partly in order to guarantee that ρ is non-singular in x for $|\zeta|=1$ (in fact it turns out to be non-singular for $\zeta \neq 0,\infty$) and partly to satisfy condition (ii) which we now discuss.

For condition (ii) the first point to establish is that $\bar{\Phi}^2 \rightarrow c^2$ as $r \rightarrow \infty$. For this we first note from (9.7) that if the Δ_r become spherically symmetric as $r \rightarrow \infty$, then the $\Delta_r \rightarrow \exp cr$ and hence, from (3.5) and (3.11) $\bar{\Phi}^2 \rightarrow c^2$ as $r \rightarrow \infty$, just as in the spherically symmetric case. It turns out that if the zeros of the polynomial $H_{\ell}(\gamma,\zeta)$ are chosen to be the first ℓ zeros of the numerator, then the relevant Δ_r, namely those for $|r| \leq \ell$, do indeed become spherically symmetric as $r \rightarrow \infty$. Thus $\bar{\Phi}^2 \rightarrow c^2$ as required. The second problem is to obtain the second term in the expression (ii). For this we use the fact that, from quite general considerations it can be shown that if $\bar{\Phi}^2 \rightarrow c^2$ then $|\bar{\Phi}|$ must become a harmonic function as $r \rightarrow \infty$, in which case we must have $|\bar{\Phi}| \rightarrow c - \kappa r^{-1}$ where κ is a constant, and the problem reduces to the determination of κ. To determine it we note from (9.5) that on the z-axis we have

$$\Delta_0(z,\rho=0) = \frac{\sinh \gamma}{H_{\ell}(z)}, \qquad \Delta_s(z,\rho=0) = 0, \quad s \neq 0, \tag{9.11}$$

for odd ℓ (and a similiar expression for even ℓ). Hence from (3.5) we have

$$\bar{\Phi}(z,0) = \frac{\partial}{\partial z} \ln \Delta_0(z,0) = \frac{\cosh cz}{\sinh cz} - \frac{H_{\ell}'(z)}{H_{\ell}(z)}. \tag{9.12}$$

Comparing (9.12) with $c - \kappa/r$ for $r = |z|$ we obtain $\kappa = \ell$. Thus the solutions (9.3) describe monopole systems of total charge $n = \ell$.

10. Properties of the Ward Family.

The first question concerning the generalized Ward solutions of the previous section is whether they describe single monopoles of charge n, or systems of separate monopoles. It turns out that they describe single monopoles. This was verified directly by Ward for $n = 2$ by showing that the Higgs field had only a single zero, and in general it can be shown by noting that the family of solutions is axially symmetric. Then one can either invoke the result of section 5 which states that axial symmetry implies single monopoles, or check from (9.9) that there are no zeros on the symmetry axis except at the origin.

In order to see that the Ward solutions are axisymmetric, one notes from (9.1) that a rotation around the z-axis in x-space can be absorbed in the phase of ζ, and then, if ρ does not contain ζ explicitly, we obtain a trivial phase-change of Δ_r after the integration in (9.5). It will be seen in section 12 that separated monopole solutions can be found by letting ρ depend explicitly on ζ.

As a matter of fact the Ward solutions are not only axisymmetric but are mirror-symmetric and symmetric with respect to the reflexion $z \to -z$. Both of these discrete symmetries can be seen from eqs. (9.1) and (9.5).

Another interesting property of the solutions (9.3) for arbitrary n is that the central functions Δ_0 of the Durham strings (the starting points for the Bäcklund transformations IB of section 3) may be obtained from the spherically symmetric ($n = 1$) solution sinh r/r by the 'translations'

$$\Delta_0(x) = (T_z)^{n-1}\left(\frac{\sinh r}{r}\right) \qquad \text{where} \qquad T_z = e^{\frac{i\pi}{2}\frac{\partial}{\partial z}} + e^{-\frac{i\pi}{2}\frac{\partial}{\partial z}} . \qquad (10.1)$$

Thus the solution for arbitrary n are generated from the n = 1 solution not by the Bäcklund transformations IB alone but by the combination of the IB transformations and the translations T_z. In fact, since T_z commutes with the Cauchy-Riemann operators (3.9) which generate the IB transformations, we

see that by the successive application of T_z and IB we obtain the two-dimensional lattice of solutions depicted in FIg. 1. However, in this lattice only the vertical strings with end-points generated by $T_z^r (IB)^{n-1}$ for r = n-1 satisfy all the conditions (i)(ii)(iii). These strings are the ones with end-points on the lines in Fig.1 which lie at 45° to the axes.

A number of different expressions can be found for the general elements Δ_r of the Durham string in (9.5). For example, by using the identity

$$\frac{\sin(\rho\cos\varphi + iz)}{(\rho\cos\varphi + iz)} = \frac{i}{2}\int_{-1}^{1} e^{-i(\rho\cos\varphi + iz)q}\, dq \qquad (10.2)$$

in (9.5) we obtain

$$\Delta_s(x) = \frac{e^{ct + is\varphi}}{4\pi}\int_{-1}^{1} e^{zq}\left(2\cos\frac{\pi q}{2}\right)^{n-1}\left(\frac{1+q}{1-q}\right)^{s/2} I_s\left(\rho\sqrt{1-q^2}\right) d \qquad (10.3)$$

where I_s is the Bessel function of the first kind. Note that in (10.3) the operator T_z^{n-1} is implemented by the function $\left(2\cos\frac{\pi q}{2}\right)^{n-1}$, and that this function serves to keep the integral finite at the end points.

A differential form of the Δ_s may be obtained from either (9.5) or (10.3), namely,

$$\begin{pmatrix}\Delta_s \\ \Delta_{-s}\end{pmatrix} = \begin{pmatrix} e^{is\varphi}\,\delta_s(\rho,z) \\ e^{-is\varphi}\,\delta_s(\rho,-z)\end{pmatrix} \qquad \text{where} \qquad \delta_s(\rho,z) = (T_z)^{n-1}\left(\frac{1-\partial_z}{\rho}\right)^s r^{2s+1}\left(\frac{d}{r\,dr}\right) \sinh r, \qquad (10.4)$$

for $s \geq 0$. Here the $\delta_s(\rho,z)$ may be generated by the function

$$e^{-uz/\rho}\sinh\left\{r\left(1+\frac{2u}{\rho}\right)^{1/2}\right\} = r\sum_{s=0}^{\infty}\frac{s^n}{n!}\,\delta_s(\rho,z). \qquad (10.5)$$

Finally an intuitive feeling for the 'shape' of the monopole can be obtained from a very interesting and elegant result on the asymptotic form of the solutions due to Prasad and Rossi[27]. These authors observed that if one makes the expansion

$$\Delta_0(x) = \sum\binom{n}{r}\frac{\sinh r(k)}{r(k)} = \frac{1}{2}\sum\frac{e^{r(k)}}{r(k)} + O\left(e^{-(r(k)+\bar{r}(k))}\right), \qquad (10.6)$$

in (8.1) then the corresponding expansion for the norm of the Higgs field takes the form

$$|\Phi| = C - \sum \frac{1}{r(k)} + O\left(e^{-(r(k)+\bar{r}(k))}\right), \tag{10.7}$$

The form (10.7) enables us to identify the asymptotic form of the Higgs field as $C - \sum r(k)^{-1}$, and to identify the 'core' of the monopole as the region where $\exp(-(r(k)+\bar{r}(k)))$ is not negligible. Because of the sharp fall off in the exponential (approximately 25 units for every unit of r) the core is very sharply defined, and a short calculation shows that it consists of the nested ellipses

$$\frac{z^2}{a^2} + \frac{\rho^2}{a^2 + k^2\pi^2} = 1 \qquad \text{where} \qquad 1 \lesssim a \lesssim \pi, \qquad |k| \leq \frac{n-1}{2}. \tag{10.8}$$

Thus the monopole core takes the shape of a discus, whose width is fixed, but whose radius increases linearly with n (see Fig. 2).

11. Positivity of the Determinant

As mentioned before, the Ansatz (7.3) yields regular gauge fields $(\vec{A}, \vec{\Phi})$ provided only that the determinant D(n) in (3.10) does not vanish. The non-vanishing of D(2) and D(3) has been demonstrated explicitly, but a rigorous proof for $n \geqslant 4$ is still lacking. (The same problem occurs in the construction of Palla et al.). However, enough progress has been made, notably by Prasad and Rossi[27], to make it virtually certain that D(n) does not vanish for general n, and we wish to briefly sketch that progress.

The first step is to note that, since the points $r(k) = 0$ in (9.7) lie within the monopole core (at $(0, \pi k)$ for $k \leq (n-1)/2$), Φ in (10.7) is non-singular outside the core. From (3.15) this implies that $\det D(n) \neq 0$ outside the core and so the problem reduces to proving that $\det D(n)$ is not zero <u>inside</u> the monopole core (as defined by (10.8)).

To investigate the situation inside the core one uses (9.5) to write the determinant in the form

$$\det D(n) = \left(\frac{e^{2t}}{2\pi}\right)^n \prod_{s=1}^{n} \int_0^{2\pi} d\psi_s\, e^{-\rho\cos\psi_s} \frac{\sin(\rho\sin\psi_s + iz)}{H(\rho\sin\psi_s + iz)} J(n,\psi), \quad \text{where } J = \begin{vmatrix} \delta(n-1) & \delta(n-2) & \cdot & \delta(0) \\ \delta(n-2) & \delta(n-3) & \cdot & \delta(-1) \\ \vdots & \vdots & & \vdots \\ \delta(1) & \delta(0) & \cdot & \delta(2-n) \\ \delta(0) & \delta(-1) & \cdot & \delta(1-n) \end{vmatrix}, \tag{11.1}$$

and $\delta(k) = \exp ik\psi_k$. Then noting that

$$\frac{\sin(\rho\sin\psi_k + iz)}{H_n(\rho\sin\psi_k + iz)} = \prod_{\frac{n+1}{2}}^{\infty} \left(1 - \frac{(\rho\sin\psi + iz)^2}{m^2\pi^2}\right), \tag{11.2}$$

and recalling[25] from the theory of unitary groups the Weyl identity,

$$J(n,\psi) = \prod_{q<s}^{n} \sin^2\left(\frac{\psi_s - \psi_q}{2}\right), \tag{11.3}$$

we see that the determinant can be written in the form

$$\det D(n) = \int d\mu(\psi)\, e^{-\rho\sum_{s=1}^{n}\cos\psi_s} \prod_{s=1}^{n} \left|\frac{\sin(\rho\sin\psi_s + iz)}{H_n(\rho\sin\psi_s + iz)}\right| \cos\theta, \tag{11.4}$$

for odd n (and a similiar expression for even n).

where $d\mu(\psi)$ is the positive measure obtained by weighting $d\psi_1 \dots d\psi_n$ with (11.3) and θ is the phase

$$\theta = \sum_{s=1}^{n} \sum_{m=\frac{n+1}{2}}^{\infty} \tan^{-1} \frac{2z\rho \sin\psi_s}{m^2\pi^2 + z^2 - \rho^2 \sin^2\psi_s} \quad . \tag{11.5}$$

From (11.3) and (11.4) it is clear that $|D(n)|$ is positive on the coordinates axes $z=0$ and $\rho=0$. Furthermore, since a good approximation to θ in (11.5) is

$$\theta \simeq \frac{2z\rho}{n\pi^2} \sum_{s=1}^{n} \sin\psi_s \leq \frac{2|z|\rho}{\pi^2} , \tag{11.6}$$

we see that it remains positive in the region

$$z\rho < \frac{\pi^3}{4} \tag{11.7}$$

surrounding the axes. Thus the determinant is not zero in the heart of the core as shown in Fig. 3.

Since the determinant does not vanish either at the heart of the core monopole core or outside the core, it is likely that it does not vanish anywhere, and, indeed, more refined arguments can be used to reduce the region of uncertainty (striped region in Fig. 3) still further.[27]

Some insight into the location of the zeros may also be obtained by letting the coordinate z become complex. For complex z the formulae (10.7) (9.11) are still valid and the core generalizes to

$$\frac{(\mathrm{Re}\,z)^2}{a^2} + \frac{\rho^2}{a^2 + (k\pi)^2 + (\mathrm{Im}\,z)^2} = 1 , \qquad 1 \leq a \leq \pi . \tag{11.8}$$

Thus, even for z complex, $|\mathrm{Re}\,z| \leq \pi$ in the core. Now for $\rho = 0$ we see from (9.11) that the only zeros of $|D(n)|$ are at $z = im\pi$, $m \geq n+1$. Since these points are quite distant from the $(\mathrm{Re}\,z, \rho)$ -plane, we then see that unless the zeros turn quickly toward the plane as ρ increases from 0 to $n\pi$ (and leave it again when $2\rho > (n+1)\pi$) the determinant will have no zeros for real z and ρ .

12. Toward Separated Monopole Solutions

Since separated monopole solutions are known to exist from Taubes result, and cannot be axially symmetric, it might be asked whether the Ward construction could be generalized to describe separated monopoles by relaxing the condition of axial symmetry. Ward has now shown that such a generalization is possible at least for two monopoles and small separations. Furthermore, the number of parameters for this case (seven) agrees with the number (4n-1) for any n predicted by the index theorem.[20]

The starting point for Ward's generalization is the denominator $H_2(\gamma)$ in (9.3). Replacing the function $H_2(\gamma)$ by the most general hermitian quadratic in ζ and ζ^{-1} one obtains

$$H_2(\gamma,\zeta) = \gamma^2 + \gamma\varepsilon + \delta^2 \qquad \begin{aligned} \varepsilon &= a + \alpha\zeta - \bar\alpha\zeta^{-1} \\ \delta^2 &= b + \beta\zeta - \bar\beta\zeta^{-1} + \sigma\zeta^2 + \bar\sigma\zeta^{-2} , \end{aligned} \tag{12.1}$$

a and b are real, and a trivial overall constant has been omitted. There are 8 real parameters in (12.1), but, as we shall see, there is a normalization condition which reduces them to seven.

Since $2\gamma = 2z + x_+\zeta - x_-\zeta^{-1}$ it is clear that the linear term in (12.1) can be eliminated by a space-translation, and that after it has been eliminated in this way (12.1) reduces to the sum of squares

$$H_2(\gamma,\zeta) = \gamma^2 + \delta^2(\zeta) \tag{12.2}$$

As a matter of fact, by using also space rotations, three of the parameters in $\delta(\zeta)$ can be eliminated. Ward has chosen to eliminate them so that

$$\delta^2(\zeta) = d^2 \left[\cos 2\alpha + \tfrac{1}{2} \sin 2\alpha (\zeta - \zeta^{-1}) \right] \tag{12.3a}$$

where d and α are real, and for reasons of symmetry it will be convenient also to eliminate them (i.e. choose the axes) in such a way that

$$\delta^2(\zeta) = d^2 \left[\cos^2\alpha - \tfrac{1}{4} \sin^2\alpha (\zeta - \zeta^{-1})^2 \right] \tag{12.3b}$$

with the same d and α . Note that the axisymmetric limit is obtained when $\sin\alpha = 0$. The general relationship between the 6 parameters of the Euclidean group and the 8 parameters in (12.1), including the reduction from (12.1) to (12.3), is given in Appendix C.

The Ansatz proposed by Ward for the separated monopole solution is the generalization of (9.10) which is obtained by replacing $H_2(\gamma)$ by the $H_2(\gamma,\zeta)$ defined in (12.2) and hence is

$$\tilde{g} = \begin{bmatrix} \dfrac{e^{f}+e^{-f}}{H_2(\gamma,\zeta)} & -e^{-f}\zeta^2 \\ e^{-f}\zeta^{-2} & H_2(\gamma,\zeta)e^{-f} \end{bmatrix} \quad \text{where } f = \left(\frac{\pi}{2}\right)\frac{\gamma}{\delta} . \tag{12.4}$$

Here $\tilde{g}$ is in the real, time-independent, gauge, and to transform it to the AW gauge, one uses the gauge-transformations

$$X(\zeta^{-1}) = \begin{pmatrix} e^{\hat{\nu}/2} & 0 \\ 0 & e^{-\hat{\nu}/2} \end{pmatrix} \qquad Y(\zeta) = \begin{pmatrix} e^{\hat{\mu}/2} & -H_2(\gamma,\zeta)\zeta^2 e^{-\hat{\mu}/2} \\ 0 & e^{-\hat{\mu}/2} \end{pmatrix}, \tag{12.5}$$

in analogy to (9.10), but where $\hat{\mu}$ and $\hat{\nu}$ are now the parts of f which are regular for $|\zeta| \lessgtr 1$. Thus

$$\hat{\mu}(\zeta) = 2\Gamma(\zeta),\ |\zeta|<1 \qquad\qquad \text{where} \quad \Gamma(\zeta) = \frac{1}{2\pi i}\int \frac{d\eta}{\eta-\zeta} f \; + \; \mathcal{U}(x) \tag{12.6}$$
$$\hat{\nu}(\zeta^{-1}) = 2\Gamma(\zeta),\ |\zeta|>1$$

and $\mathcal{U}(x)$ is a ζ-independent function of x . Then g takes the form

$$g = X\tilde{g}Y = \begin{pmatrix} \rho & -\zeta^2 \\ \zeta^{-2} & 0 \end{pmatrix}, \quad \text{where} \quad \rho = \frac{e^{\hat{\mu}}+e^{\hat{\nu}}}{H_2(\gamma,\zeta)} = \frac{e^{f}+e^{-f}}{H_2(\gamma,\zeta)}\, e^{\frac{\hat{\mu}+\hat{\nu}}{2}} . \tag{12.7}$$

The function ρ in (12.7) will generate solutions of the Bogomolny equation, of course, only if ρ , and hence $\hat{\mu}$ and $\hat{\nu}$, are functions of the twistor coordinates μ, ν, ζ only. The condition for this is

$$(P_\mu \partial_\mu)\hat{\mu} = (P_\mu \partial_\mu)\hat{\nu} = 0 \tag{12.8}$$

where the P_μ are the operators defined in (7.10). On applying (12.8) to (12.6) and using the fact that γ already satisfies (12.8), one finds that

the denominator $(\eta-\zeta)$ in (12.7) is cancelled, and one obtains

$$(P_\mu \partial_\mu)\mathcal{U}(x) = \frac{-1}{2\pi i}\oint \frac{d\eta}{\eta^2\delta}\begin{pmatrix}1\\ \eta\end{pmatrix} . \tag{12.9}$$

It is easy to see that (12.9) will be satisfied if, and only if,

$$(-4\pi i)\,\mathcal{U}(x) = \bar{v}\int \frac{d\eta}{\eta}\frac{1}{\delta} + u\int\frac{d\eta}{\delta} , \quad \text{where} \quad \begin{matrix} u = x+iy \\ \bar{v} = z - ix_4 \end{matrix} . \tag{12.10}$$

Thus the condition (12.8) is satisfied and $\mathcal{U}(x)$ is determined at one stroke.

Since the time coordinate enters in (12.7) only through $\mathcal{U}(x)$ and since, from (9.7) and the boundary condition $\Phi^2 \to c^2$, the time dependence of ρ must be of the form exp(ct), we see that Q must satisfy the normalization condition

$$\frac{1}{2\pi i}\oint \frac{d\eta}{\eta}\,\frac{1}{\delta(\eta)} = c . \tag{12.11}$$

This is the normalization condition referred to earlier, and if we use the form (12.3) for $\delta(\zeta)$, it reduces to

$$d = \frac{1}{2\pi c}\int_0^{2\pi} \frac{d\psi}{(\cos 2\alpha + i\sin 2\alpha \sin\psi)^{1/2}} = \frac{1}{2\pi c}\int_0^{2\pi} \frac{d\psi}{(\cos^2\alpha + \sin^2\alpha \sin^2\psi)^{1/2}} , \tag{12.12}$$

thus reducing the 2 parameters d and α to one.

Since from (12.7)(12.12) it is clear that the Ansatz (12.4) describes a real time-independent, solution of the Bogomolny equations, it remains only to verify that the solution is regular, satisfies the boundary condition and describes separated monopoles. The verification of the first two points depends essentially on the fact the Ansatz (12.4) is obtained by a continuous deformation of the parameter α from the axisymmetric Ansatz (9.2). Using this fact one easily verifies that the boundary condition is satisfied for all α . Similarly since ρ in (12.5) is regular in $\vec{x}$, and the determinant $\Delta_0^2 - \Delta_1\Delta_{-1}$ does not vanish in the axisymmetric limit, one sees that the solution will be regular for small deformations (i.e. small separations) at least. A proof for large separations is still lacking, however.

Finally, one has to establish that the Ansatz (12.4) does actually describe separated system i.e. the Higgs field has separated zeros. For this purpose it is convenient to note that the Ansatz (9.3) is symmetric with respect to rotations through an angle π around the coordinate axes, provided that the coordinate axes are suitably chosen. To see this we use the result of Appendix C to implement these rotations by the transformations $(\zeta \to \zeta^{-1})$, $(\zeta \to -\bar{\zeta}^{-1})$ and $(\zeta \to -\zeta)$ in ζ-space. We then see that (12.3a) is invariant with respect to the 2nd transformation $(\zeta \to -\zeta^{-1})$ while (12.3b) is invariant with respect to all three. Thus if we choose axes so that δ takes the form (12.3b) we have invariance with respect to π-rotations around each of the coordinate axes. (This is why we have preferred the form (12.3b) for δ to Ward's original form).

Once it is established that the Ansatz is invariant with respect to π-rotations around the three axes, it follows from the fact that there are only two monopoles (zeros of the Higgs field $\Phi(x)$) that they must lie on one of the coordinate axes, and, if they are not separated, must lie at the origin. Hence to show that the zeros are separated it suffices to show that $\Phi(0) \neq 0$. A computation shows that indeed $\Phi(0) \neq 0$, unless $\sin\alpha = 0$. Hence for <u>all</u> values of the parameter $\sin\alpha$, except zero, the Ansatz (12.4) (if regular) describes separated monopoles. Note that, by continuity, the axis on which the monopoles are located must be the same for all $\sin\alpha$, and hence to determine it one need only compute the zeros for small $\sin\alpha$. Using the results of Ward for small $\sin\alpha$, it is easy to see that the axis in question is the x-axis. Thus, for any $\sin\alpha \neq 0$, the monopoles are separated and lie on the x-axis.

13. A Proposal for an n-Separated-Monopole Solution.

Just as these notes were being completed we received a preprint [25] proposing a generalization of the n = 2 separated Ansatz (12.4) to arbitrary n. The proposal admits (4n-1) independent parameters, as predicted by the index theorem [20].

The essence of the proposal is to replace the polynomials H_2 and f in (12.4) by the polynomials

$$H_n(\gamma,\zeta) = \sum_{s=0}^{n} a_s(\zeta)\gamma^s \qquad \text{and} \qquad f_n = 2\pi i \sum_{s=1}^{n} n_s \prod_{r\neq s}\left(\frac{\gamma-\gamma_r}{\gamma_s-\gamma_r}\right), \tag{13.1}$$

respectively, where the $a_s(\zeta)$ are polynomials of degree n-s in ζ and ζ^{-1} satisfying the reality condition $a_s(\zeta) = \bar{a}_s(-\bar{\zeta}^{-1})$, the $\gamma_s(\zeta)$ are the roots of γ in $H_n(\gamma,\zeta)$, and the n_s are integers (or half-odd-integers for odd n). It is evident that for suitable choice of the integers n_s the zeros of H_n will be cancelled by the zeros of the numerator in (12.4), and the integers chosen are the smallest with this property. Note that the number of real parameters in (13.1) is the number in H_n and this is just $n(n+2)$.

The condition that there should exist a gauge-transformation from (12.4) to the AW-gauge (8.1) is exactly the same as in the n = 2 case, namely that there should exist a $\hat{\mu}$ and a $\hat{\nu}$ satisfying (12.8), (with f replaced by f_n). Accordingly, in analogy to (12.9) the condition can be written as

$$(P_\mu \partial_\mu)\, U(x) = \oint \frac{d\eta}{\eta}\left(\frac{\partial f_n}{\partial\gamma}\right)\begin{pmatrix}1\\ \eta\end{pmatrix}. \tag{13.2}$$

However, in contrast to the n = 2 case, eq. (13.2) can be integrated, to give

$$(-4\pi i)\, U(x) = \bar{v}\int \frac{d\eta}{\eta}\left(\frac{\partial f_n}{\partial\gamma}\right) + k(u), \quad \text{where} \quad \frac{\partial k(u)}{\partial u} = \int d\eta\left(\frac{\partial f_n}{\partial\gamma}\right), \tag{13.3}$$

if, and only if, the integrability conditions

$$\oint \frac{d\eta}{\eta}\,\eta^r\,\frac{\partial^2 f}{\partial\gamma^2} = 0, \qquad r = 0, \pm 1, \tag{13.4}$$

are satisfied. Note that these conditions are equivalent to the condition that the coefficient of $\bar{v} = z - ix_4$ in (13.3) be constant.

By expanding f_n in powers of γ one sees that the integrability conditions may be written as

$$\oint \frac{d\eta}{\eta}\, \eta^q\, b_p(\eta) = 0 \qquad \begin{matrix} |q| < p \\ 2 \le p \le n-1 \end{matrix} \qquad f_n(\gamma,\eta) = \sum_{r=0}^{n-1} b_r(\eta)\,\gamma^r \tag{13.5}$$

and it is then evident that there are n(n-2) conditions altogether. Together with the normalization condition from (13.3) these conditions reduce the number of independent parameters to (4n-1).

The conditions (13.5) can also be stated as the condition that f_n must be of the form

$$f_n(\xi,\gamma) = \sum_{i=0}^{n-1}\sum_{j=-i}^{i} a_{ij}\, \gamma^i\, \xi^j \qquad a_{i,\pm j} = 0 \quad |j| \le i+1 \tag{13.6}$$

and the compatibility of (13.6) and (13.1) yield the n(n-2) conditions on the parameters in algebraic form.

We still have to consider, of course, whether the proposed solution is regular, satisfies the boundary conditions and describes n separated monopoles. That the boundary condition will be satisfied is guaranteed by the Yukawa equation (9.7) and the exponential time-dependence in (12.7) and may be verified explicitly by noting that the integrability conditions (13.4) (which incorporate the time-independence of $f(\xi,\gamma)$) limit the exponential growth of the computed from (12.7) to be of order one. The regularity of the solution depends, as usual, on the non-vanishing of the Durham determinant and, if the determinant obtained in the axisymmetric limit really is non-zero, then the small-separation argument used for n = 2 should again be valid. Finally, the fact that there is the full complement of parameters allowed by the index theorem suggests that the monopoles can indeed be separated. Thus there is strong prima facie evidence that the proposal (13.1) is correct.

Appendix A.

We wish to show that for static fields generated by the Durham string, we have [22]

$$\Phi^2 = c^2 - \Delta \ln(\det D), \tag{A1}$$

and that if the system is axially symmetric, we have

$$(\Phi,\omega) = \partial_z \ln(\det D) \quad \text{and} \quad \omega^2 + (c^2 - \Phi^2) = 2\rho\,\partial_\rho \ln(\det D). \tag{A2}$$

Here it is assumed that since the gauge potentials are static, the string-functions $\Delta_r(x)$ have only exponential time-dependence

$$\Delta_r(x) = \Delta_r(\vec{x})\, e^{ct}, \tag{A3}$$

and that the axial symmetry condition (5.1) may be reduced to the standard form [24]

$$\partial_\varphi \Phi = n\sigma_3 \Phi \qquad \text{where} \qquad \omega = n\sigma_3 + A_\varphi, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A4}$$

and n is the topological charge. The reason that $\omega - A_\varphi$ can be reduced to the constant vector $n\sigma_3$ is that ω is smooth and D_φ has only integer eigenvalues. (The identification of the integer with the topological charge comes from the formula (2.5)). It will also be convenient to write the formula (3.5) for the components A_4 and A_φ of the gauge-potential as

$$A_4 = \frac{-1}{2f}\begin{bmatrix} f_{,z} & e_{,u} \\ g_{,\bar{u}} & -f_{,3} \end{bmatrix} \qquad A_\varphi = \frac{-\rho}{2f}\begin{bmatrix} f_{,\rho} & e^{i\varphi} e_{,\bar{v}} \\ e^{-i\varphi} g_{,v} & -f_{,\rho} \end{bmatrix}. \tag{A5}$$

To establish (A1) we note from (A5) that

$$\Phi^2 = \mathrm{tr}\, A_4^2 = \frac{1}{4f^2}\left[2f_{,3}^2 + e_{,u} g_{,\bar{u}}\right], \tag{A6}$$

Hence under the Bäcklund transformation B in (3.7) we have

$$\Phi^2 \to \frac{1}{4f^2}\left[2f_{,3}^2 - e_{,\bar{v}} g_{,v}\right] = \Phi^2 - \Delta \ln f \tag{A7}$$

where in the second step we have used the Yang equation (3.3). Furthermore, since the fields are static, Φ^2 is invariant with respect to the gauge transformations I. Hence (A7) holds also for the combined transformation BI. Iterating the BI transformation n times we then have

$$\Phi^2 = c^2 - \Delta \ln (f_n f_{n-1} \cdots f_2 f_1) . \tag{A8}$$

But from (3.9) and (3.10) we have, by definition

$$f_r = \det D(r) \,/\, \det D(r-1) . \tag{A9}$$

Inserting (A9) into (A8) we obtain (A1) as required.

To establish (A2) we note first that the invariants ω^2 and (ω, Φ) are given by

$$\omega^2 = \mathrm{tr}(A_\varphi + n\sigma_3)^2 \qquad \text{and} \qquad (\omega, \Phi) = \mathrm{tr}(A_\varphi + n\sigma_3)A_4 \;, \tag{A10}$$

where, from (A5),

$$\mathrm{tr}(A_\varphi + n\sigma_3)^2 = \frac{\rho^2}{2f^2}\left[f_\rho^2 + e_{\bar v} g_v\right] - \frac{2n}{f} f_\rho + 2n^2$$

and

$$\mathrm{tr}(A_\varphi + n\sigma_3)A_4 = \frac{\rho}{4f^2}\left[2 f_\rho f_z + e_u g_v + e_{\bar v} g_{\bar u}\right] - \frac{n}{f} f_z . \tag{A11}$$

Under the Bäcklund transformations B the integer n changes to $-(n-1)$ (the minus because e and g interchange positions). Taking the change in n and the Yang equations (3.3) into account, we find from (A11) and (A10) that

$$\omega^2 \to \omega^2 + (2\rho\partial_\rho - \Delta)\ln f \qquad \text{and} \qquad (\omega, \Phi) \to (\omega, \Phi) + \partial_z \ln f \;, \tag{A12}$$

under the transformations B and BI. Iterating the BI transformation n times and using (A9) we obtain (A2) as required.

Appendix B

To show that the planes defined by $\omega = x\pi$ are self-dual consider any displacement dx in such a plane. Since dx belongs to the $D(\frac{1}{2}\,\frac{1}{2})$ representation of SO(4) it can be written as the direct product $\bar\eta \times \lambda$ of two 2-spinors. Then the condition $dx\,\pi = 0$ that dx lies in the plane takes the form

$$\bar\eta \times (\lambda \wedge \pi) = 0 , \tag{B1}$$

where wedge denotes outer product in the 2-space. Since (B1) implies that λ is parallel to π, we see that displacements dx in the plane are just those which are of the form

$$dx = \bar\eta \times \pi , \tag{B2}$$

where η is free and π is fixed. Let us now consider the anti-symmetric product of two such displacements dx, dx'. Normally such products belong to the D(10) $\oplus$ D(01) representation of SO(4), but because π is fixed we obtain

$$(\bar\eta \wedge \bar\eta')(\pi . \pi') + (\bar\eta . \bar\eta')(\pi \wedge \pi) = (\bar\eta \wedge \bar\eta')(\pi . \pi') , \tag{B3}$$

which manifestly belongs only to the D(10) representation. But belonging to the D(10) part of D(10)$\oplus$ D(01) is just the definition of anti-self-duality. Hence the planes $\omega = x\pi$ are anti-self-dual, as required.

Appendix C

In order to obtain the connection between the parameters of the Euclidean group and those of the 2-form (12.1) we first note from (7.1) that $(\pi_1, \pi_2$ is a translational scalar and rotational 2-spinor. It follows that the quantity

$$\vec{\Sigma} = \tilde{\pi} C \vec{\sigma} \pi \qquad \text{where} \qquad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{C1}$$

tilde denotes transpose, and $\vec{\sigma}$ are the Pauli matrices, is a translational scalar and rotational 3-vector. On the other hand, if we recall that $\zeta = \pi_1/\pi_2$ it is easy to verify that the 2-form (12.1) may be written in the homogeneous form

$$H_2(\gamma,\zeta) = \left\{ (x\cdot\Sigma)^2 + 2(x\cdot\Sigma)(g\cdot\Sigma) + \frac{d^2}{4}(e\cdot\Sigma)(e'\cdot\Sigma) \right\} / \Sigma_3^2 , \tag{C2}$$

where d^2 is a positive constant, g, e, e', are real constant 3-vectors (with e, e' normalized to unity), and these 8 parameters are related to the original 8 parameters in (12.1) in an obvious manner.

By inspection of (C2) one sees that the vector g may be eliminated by a space translation and the vectors e, e' rotated to any position so long as the angle between them remains fixed. Thus the Euclidean invariant parameters in (C2) are d^2 and the inner-product (e.e').

When g has been eliminated and $\{e, e'\}$ brought to the positions $\{(0\,0\,1), (\sin 2\alpha, 0, \cos 2\alpha)\}$ and $\{(\sin\alpha, 0, \cos\alpha), (-\sin\alpha, 0, \cos\alpha)\}$ the expression $H_2(\gamma,\zeta)$ reduces to

$$H_2(\gamma,\zeta) = \left\{(x\cdot\Sigma)^2 + \frac{d^2}{4}\Sigma_3(\Sigma_3\cos 2\alpha + \Sigma_1 \sin 2\alpha)\right\}/\Sigma_3'^2 = \left\{\gamma^2 + d^2\left[\cos 2\alpha + \tfrac{1}{2}\sin 2\alpha(\zeta-\zeta^{-1})\right]\right\}\left(\frac{\Sigma_3}{\Sigma_3'}\right)^2 \tag{C3a}$$

and

$$H_2(\gamma,\zeta) = \left\{(x\cdot\Sigma)^2 + \frac{d^2}{4}(\Sigma_3^2\cos^2\alpha - \Sigma_1^2\sin^2\alpha)\right\}/\Sigma_3'^2 = \left\{\gamma^2 + d^2\left[\cos^2\alpha - \tfrac{1}{4}\sin^2\alpha(\zeta-\zeta^{-1})^2\right]\right\}\left(\frac{\Sigma_3}{\Sigma_3'}\right)^2 \tag{C3b}$$

respectively, These are just the expressions (12.2) (12.3) except for the factors

$$\left(\frac{\Sigma_3}{\Sigma_3'}\right)^2 = \left(\frac{\pi_1}{\pi_1'}\right)^2\left(\frac{\pi_2}{\pi_2'}\right)^2 , \tag{C4}$$

where

$$\frac{\pi_1'}{\pi_1} = \cos\beta_2 e^{i(\beta_1+\beta_3)} + \zeta^{-1}\sin\beta_2 e^{i(\beta_1-\beta_3)} , \qquad \frac{\pi_2'}{\pi_2} = \cos\beta_2 e^{-i(\beta_1+\beta_3)} - \zeta\sin\beta_2 e^{i(\beta_3-\beta_1)} \tag{C5}$$

and the β's are half the Euler angles for the required rotations. But since the terms in (C4) are analytic for $|\zeta|>1$ and $|\zeta|<1$ respectively, the factors $(\Sigma_3/\Sigma_3')^2$ can be gauge to zero by the gauge transformations

$$X(\zeta^{-1}) = \operatorname{diag}\left(\frac{\pi_1}{\pi_1'}, \frac{\pi_1'}{\pi_1}\right) , \qquad Y(\zeta) = \operatorname{diag}\left(\frac{\pi_2}{\pi_2'}, \frac{\pi_2'}{\pi_2}\right) . \tag{C6}$$

Thus finally H_2 can be reduced to the simple forms

$$H_2(\gamma,\zeta) = \gamma^2 + d^2(\cos 2\alpha + \tfrac{1}{2}(\zeta-\zeta^{-1})\sin 2\alpha) , \qquad H_2(\gamma,\zeta) = \gamma^2 + d^2(\cos^2\alpha - \tfrac{1}{4}(\zeta-\zeta^{-1})^2\sin^2\alpha) ,$$

respectively, as anticipated in (12.3).

It is clear from this discussion that for any function of γ and ζ, the spacial rotations may be implemented by a suitable transformation of ζ. In particular, the rotations through an angle π about the x, y, and z axes used in section 12 may be implemented by $\zeta \to \zeta^{-1}$, $\zeta \to -\zeta^{-1}$ and $\zeta \to -\zeta$ respectively.

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DIAGRAMS

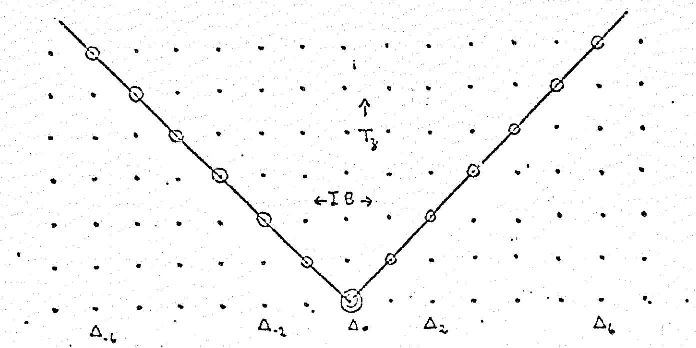

Fig. 1. Plot of the Durham strings Δ_r, $-l \le r \le l$ generated from the spherically symmetric solution $\Delta_0 = \sinh(r)/r$ (doubly circled) by successive operations of the 'translation' T_z and the Bäcklund transformations IB. The strings are drawn horizontally and only those whose ends are circled satisfy the reality and boundary conditions.

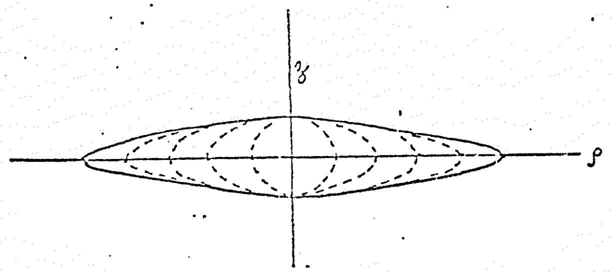

Fig. 2. Plot of the Monopole Core for Charge $Q = 5$, showing the nested ellipses as dotted curves.

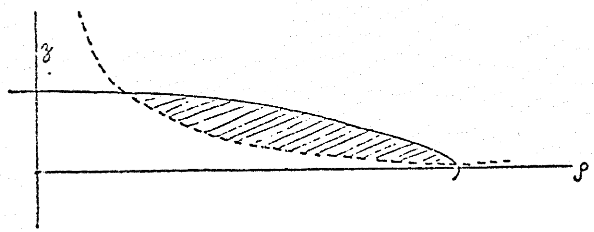

Fig. 3. Positive Quadrant of the Monopole Core with curve $z\rho = \pi^3/4$ shown as dotted line. The solutions have been shown to be non-singular outside the core and inside the dotted curve, leaving only the shaded area open to any doubt.